Syllabus of 401(iii): Simplicial Homology Theory
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Theorem 4.7.2. (No-retraction Theorem).

Theorem 4.7.3. (Brouwer's Fixed-Point Theorem).

Definition 4.8.3. (Degree of a Map)

Theorem 4.8.5. (Hopf's Classification Theorem).

Theorem 4.8.8.(The degree of antipodal map on n-sphere)

For any query, contact on the following E-mail ids:

1. **hemantksingh@maths.du.ac.in**

2. **hksinghdu@gmail.com**

Theorem 4.7.1. (Invariance of dimension). *If $m \neq n$, then*

(i) $\mathbb{S}^m$ *is not homeomorphic to* $\mathbb{S}^n$*, and*

(ii) $\mathbb{R}^m$ *is not homeomorphic to* $\mathbb{R}^n$*.*

Proof. (i) Suppose, on the contrary, there is a homeomorphism $f\colon \mathbb{S}^m \to \mathbb{S}^n$. Let $h\colon |K| \to \mathbb{S}^m$ and $k\colon |L| \to \mathbb{S}^n$ be triangulations of $\mathbb{S}^m$ and $\mathbb{S}^n$, respectively. We have already computed the homology groups of K and L. Since f is a homeomorphism, the map $g = k^{-1}fh\colon |K| \to |L|$ is also a homeomorphism. Let $g^{-1}\colon |L| \to |K|$ be its inverse. We consider the induced homomorphisms $g_*\colon H_m(K) \to H_m(L)$ and $g_*^{-1}\colon H_m(L) \to H_m(K)$ in simplicial homology. Since $g^{-1}g = I_{|K|}$, we find by theorem 4.6.4, that

$$g_*^{-1}g_* = (g^{-1}g)_* = (I_{|K|})_* = I_{H_m(K)}.$$

Similarly, we find that $g_*g_*^{-1}$ is also identity on $H_m(L)$. Therefore, $g_*\colon H_m(K) \to H_m(L)$ is an isomorphism. But this is a contradiction, because $H_m(K) = \mathbb{Z}$ whereas $H_m(L) = 0$, since $m \neq n$. This proves the theorem.

(ii) Again, suppose $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$. Since these are locally compact Hausdorff spaces, their one-point compactifications, viz., $\mathbb{S}^m$ and $\mathbb{S}^n$ must also be homeomorphic, which is a contradiction to (i) proved above. Hence, if $m \neq n$, $\mathbb{R}^m$ cannot be homeomorphic to $\mathbb{R}^n$. ∎

If $\mathbb{D}^n$ denotes the n-dimensional disk (closed), then its boundary is homeomorphic to the $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$. Thus, $\mathbb{S}^{n-1}$ is a compact subset of $\mathbb{D}^n$. Recall that a subspace A of a space X is said to be **retract** of X if there is a continuous map $r\colon X \to A$ such that $r(a) = a$ for $a \in A$. Now, one can ask the question: Is $\mathbb{S}^{n-1}$ is a retract of $\mathbb{D}^n$? If $n = 1$, this is clearly impossible because $D^1 = [-1, 1]$ is connected whereas $\mathbb{S}^0 = \{-1, 1\}$ is disconnected. If $n \geq 2$, then also the answer to the above question is "no". We have

Theorem 4.7.2. (No-retraction Theorem). *The sphere* $\mathbb{S}^{n-1}$ *cannot be a retract of* $\mathbb{D}^n$*, for any* $n \geq 1$*.*

Proof. If possible, suppose there is a retraction $r\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$. Let $i\colon \mathbb{S}^{n-1} \to \mathbb{D}^n$ be the inclusion map. Then, clearly, $ri = I_{\mathbb{S}^{n-1}}$. The case when $n = 1$ is clear because $\mathbb{D}^1$ is connected whereas $\mathbb{S}^0$ is not. Hence we assume that $n > 1$. Let $h\colon |K| \to \mathbb{D}^n$ and $k\colon |L| \to \mathbb{S}^{n-1}$ be triangulations of the disc and sphere. Notice that we have continuous maps $k^{-1}rh\colon |K| \to |L|$ and $h^{-1}ik\colon |L| \to |K|$ such that their composite is identity, i.e., $(k^{-1}rh)(h^{-1}ik) = I_{|L|}$. This means the composite

$$\mathbb{Z} = H_{n-1}(L) \to H_{n-1}(K) \to H_{n-1}(L) = \mathbb{Z}$$

is identity on $\mathbb{Z}$ by theorem 4.6.4. But this last map factors through the zero group $H_{n-1}(K) = 0$, which is a contradiction. ∎

Recall that a space X is said to have fixed-point property if for every continuous map $f\colon X \to X$, there exists a point $x_0 \in X$ such that $f(x_0) = x_0$. From the first course in real analysis, we know that the unit interval [0,1] has the fixed-point property. The following important theorem is a far reaching generalization of this result.

Theorem 4.7.3. (Brouwer's Fixed-Point Theorem). *Let $f\colon \mathbb{D}^n \to \mathbb{D}^n$, $n \geq 1$ be any continuous map. Then f has at least one fixed point.*

Proof. Suppose f has no fixed points. This means for every x, $f(x) \neq x$. Now, we define a map $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ as follows: For any $x \in \mathbb{D}^n$, consider the line segment (vector) joining $f(x)$ to x and extend it. Then, this vector when produced will meet $\mathbb{S}^{n-1}$ exactly at one point, which we call $g(x)$ (see Fig. 4.9). First, we prove that $g\colon \mathbb{D}^n \to \mathbb{S}^{n-1}$ is continuous. Note that for each x, the nonzero vector $x - f(x)$ can be multiplied by a unique positive scalar, say λ, such that $g(x) = f(x) + \lambda(x - f(x))$, and this λ depends on x. Since $g(x)$ lies on $\mathbb{S}^{n-1}$, its norm is 1 and so $||f(x) + \lambda(x - f(x))|| = 1$. This implies that

$$||f(x)||^2 + \lambda^2||(x - f(x))||^2 + 2\lambda f(x) \cdot (x - f(x)) = 1.$$

This is a quadratic equation in λ having only one positive real root λ. Therefore, by the formula for the roots of a quadratic equation, we get $\lambda = \frac{-f(x)\cdot(x-f(x))}{||x-f(x)||^2}$. This proves that λ is a continuous function of x and therefore g is a continuous function. But g is clearly a retraction of $\mathbb{D}^n$ onto $\mathbb{S}^{n-1}$, a contradiction to the no-retraction theorem. ■

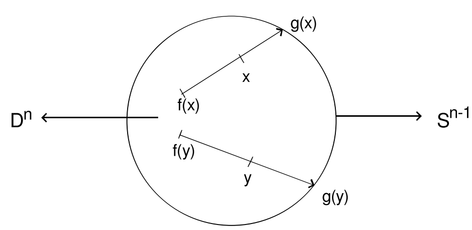

Fig. 4.9: The map g becomes a retraction

The following result gives an interesting application of the Brouwer fixed-point theorem to a result of linear algebra. We have

Proposition 4.7.4. *Let A be an $n \times n$ real matrix with positive entries. Then A has a positive eigen value.*

Proof. Consider the Euclidean space $\mathbb{R}^n$, and observe that A determines a linear transformation from $\mathbb{R}^n$ to itself. Also, note that if $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, where $x_i \geq 0$ for all $i = 1, \ldots, n$, then, since all entries of A are positive, Ax also has the same property, i.e., A maps the positive octant P including its boundary to itself. Furthermore, if one entry of x is positive, then all entries of Ax are positive. Let us also point out that if $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, then $\mathbb{S}^{n-1} \cap P$ is homeomorphic to the $(n-1)$-disk D^{n-1}. Now, we can define a map $f\colon \mathbb{S}^{n-1} \cap P \to \mathbb{S}^{n-1} \cap P$ by putting $f(x) = Ax/\|Ax\|$. Then evidently f is continuous. Hence, by the Brouwer's fixed point theorem, there exists a non-zero vector $x_0 \in \mathbb{S}^{n-1} \cap P$ such that $f(x_0) = x_0$, i.e., $Ax_0/\|Ax_0\| = x_0$. This says that $Ax_0 = \|Ax_0\|x_0$ which means $\|Ax_0\|(\neq 0)$ is an eigen value of A. ■

Knowing that every continuous map $f\colon X \to X$ has a fixed point is an important property of the space X, and it has interesting applications. Let us illustrate this by another example. Suppose we have a set of continuous functions $f_i\colon \mathbb{R}^n \to \mathbb{R}, i = 1, 2, \ldots, n$, each of which has a nonempty zero set, i.e., there exists some points of $\mathbb{R}^n$ where f_i is zero. Now, the question is: Is there a common zero of all these functions f_i? In other words, do the following system of simultaneous equations has a solution in $\mathbb{R}^n$:

$$\begin{aligned} f_1(x_1, x_2, \ldots, x_n) &= 0 \\ f_2(x_1, x_2, \ldots, x_n) &= 0 \\ &\vdots \\ f_n(x_1, x_2, \ldots, x_n) &= 0 \end{aligned}$$

This question is really a question whether the space $\mathbb{R}^n$ has the fixed-point property. To see why this is so, let us consider the following continuous maps $g_i\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$g_i(x_1, x_2, \ldots, x_n) = f_i(x_1, \ldots, x_n) + x_i$$

for $i = 1, 2, \ldots, n$. We consider the map $h\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h(x_1, \ldots, x_n) = (g_1(x_1, \ldots, x_n), \ldots, g_n(x_1, \ldots, x_n)).$$

Notice that h is continuous because each g_i is continuous. Now, observe that h has a fixed point $(a_1, \ldots, a_n) \in \mathbb{R}^n$ if and only if for each $i = 1, \ldots, n$, we have

$$g_i(a_1, \ldots, a_n) = a_i,$$

and that will happen if and only if

$$f_1(a_1, \ldots, a_n) = 0 = f_2(a_1, \ldots, a_n) = \ldots = f_n(a_1, \ldots, a_n),$$

i.e., all the $f_i's$ have a common zero. It may be remarked that the Euclidean space $\mathbb{R}^n$ used in this example does not have the fixed-point property because translations by nonzero vectors do not have fixed points. On the other hand, by Brouwer's theorem, the cube I^n, where $I = [-1, 1]$, has the fixed-point property and so can very well be used for the space X to conclude that the simultaneous equations have a solution.

Exercises

1. If a space X has the fixed-point property, then show that any space Y which is homeomorphic to X also has the fixed-point property.
2. If A is a retract of X and if X has the fixed-point property, then show that A also has the fixed point property.
3. Let X be a compact metric space and $f\colon X \to X$ be a fixed-point free map. Prove that there is an $\epsilon > 0$ such that $d(x, f(x)) > \epsilon$ for all $x \in X$.
4. By giving concrete examples prove that the 2-sphere, the torus and the Klein bottle do not have the fixed-point property.
5. Prove that the projective plane $\mathbb{P}^2$ has the fixed-point property (this may be bit difficult at this stage!).
6. If a polyhedron $A \subset X$ is a retract of a polyhedron X, then show that for $q \geq 0,\ H_q(A)$ is a direct summand of $H_q(X)$.
7. Prove that an injective map between two polyhedra does not necessarily induce an injective map between their homology groups.
8. Show that the following conditions are equivalent:
   (a) $\mathbb{S}^{n-1}$ is not a retract of $\mathbb{D}^n$.
   (b) $\mathbb{D}^n$ has the fixed-point property.
   (c) The n-simplex δ_n has the fixed-point property.
9. Show that the map $g\colon \mathbb{RP}^3 \to \mathbb{RP}^3$ defined by the linear transformation $T\colon \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$ does not have the fixed-point property. More generally, show that $\mathbb{RP}^{2k+1}$ does not have the fixed-point property.
10. Prove that the homology groups of the real projective spaces $\mathbb{RP}^n,\ n \geq 2$, and that of the complex projective spaces $\mathbb{CP}^n,\ n \geq 1$ are given by the following (See Examples 1.1.4 (a) and 1.1.4 (b). The cases $n \geq 3$ need concepts not covered so far.)

    (i) n is odd:

    $$H_q(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & q = 0, n \\ \mathbb{Z}_2, & 0 < q < n, q \text{ odd} \\ 0, & \text{otherwise,} \end{cases}$$

    (ii) n is even:

    $$H_q(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}_2, & 0 < q < n, q \text{ odd} \\ 0, & \text{otherwise,} \end{cases}$$

(iii) The complex case

$$H_q(\mathbb{C}P^n) \cong \begin{cases} \mathbb{Z}, & q \text{ is even} \\ 0, & \text{otherwise,} \end{cases}$$

11. Show that any nonsingular linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ defines a continuous map $g\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ which has a fixed point. More generally, prove that $\mathbb{RP}^{2k}$ has the fixed-point property for any $k \geq 0$.

12. Prove that any compact locally contractible space has the fixed-point property.

4.8 Degree of a Map and its Applications

Recall that a homomorphism $f\colon \mathbb{Z} \to \mathbb{Z}$ of the infinite cyclic group is completely determined by the image of its generator $1 \in \mathbb{Z}$ under the map f, i.e., f is simply multiplication by the integer $f(1) = n$. This fact is used in the following:

Definition 4.8.1. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n, (n \geq 1)$ be a continuous map and $h\colon |K| \to \mathbb{S}^n$ be any triangulation of $\mathbb{S}^n$. Then we know that f induces a homomorphism $(h^{-1}fh)_*\colon H_n(K) \to H_n(K)$. Since $H_n(K) \cong \mathbb{Z}$, there is a unique integer d such that for every element $\alpha \in H_n(K)$, $(h^{-1}fh)_*(\alpha) = d\alpha$. This unique integer d is called the* **degree** *of f and we denote it by $\deg f$.*

We must prove that the integer d does not depend on the chosen triangulation $h\colon |K| \to \mathbb{S}^n$. For this, let $k\colon |L| \to \mathbb{S}^n$ be another triangulation of $\mathbb{S}^n$. Note that $\phi = k^{-1}h\colon |K| \to |L|$ and $\phi^{-1} = h^{-1}k\colon |L| \to |K|$ are homeomorphisms. Therefore, $(k^{-1}fk)_*(\alpha) = (k^{-1}h)_*(h^{-1}fh)_*(h^{-1}k)_*(\alpha) = \phi_*(h^{-1}fh)_*\phi_*^{-1}(\alpha) = \phi_*(d \cdot (\phi^{-1})_*(\alpha)) = \phi_*(\phi^{-1})_*(d \cdot \alpha) = d \cdot \alpha$, which says that $(k^{-1}fk)_*$ is again multiplication by d. This proves the claim.

The following result is a consequence of the definitions.

Proposition 4.8.2. (a) *The identity map $I_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$ has degree $+1$.*

(b) *If $f\colon \mathbb{S}^n \to \mathbb{S}^n$, $g\colon \mathbb{S}^n \to \mathbb{S}^n$ are continuous maps, then* $\deg(g \circ f) = \deg g \cdot \deg f$.

(c) *The degree of any homeomorphism is ± 1.*

Proof. (a) This follows from the fact that the identity map $I_{\mathbb{S}^n}$ induces the identity map in homology.

(b) Let $k\colon |K| \to \mathbb{S}^n$ be a triangulation of $\mathbb{S}^n$. Suppose $\deg f = n_1$, $\deg g = n_2$. Then for any $\alpha \in H_n(K)$, we have

$$\begin{aligned}(k^{-1}g \circ fk)_*(\alpha) &= (k^{-1}gk)_*(k^{-1}fk)_*(\alpha)) \\ &= (k^{-1}gk)_*(n_1(\alpha)) \\ &= n_2(n_1\alpha) \\ &= (n_2n_1)\alpha.\end{aligned}$$

Hence, $\deg(g \circ f) = \deg g \cdot \deg f$.

(c) Let $h\colon \mathbb{S}^n \to \mathbb{S}^n$ be a homeomorphism. Then $h^{-1} \circ h = I_{\mathbb{S}^n}$ and hence, by (a), we have

$$\deg(h^{-1} \circ h) = 1 = \deg h^{-1} \cdot \deg h.$$

Since $\deg h, \deg h^{-1}$ both are integers, we must have $\deg h = \deg h^{-1} = +1$ or -1. ■

Next, we are going to prove that if $f, g\colon \mathbb{S}^n \to \mathbb{S}^n$ are two homotopic maps then $\deg f = \deg g$. First, observe that if $h\colon |K| \to \mathbb{S}^n$ is any triangulation of $\mathbb{S}^n$, then $h^{-1}fh,\ h^{-1}gh\colon |K| \to |K|$ are also homotopic. We will prove later (see Theorem 4.8.4) that two homotopic maps induce identical homomorphism in homology. Hence the result follows at once. However, proving that homotopic maps induce identical homomorphisms in homology is much more involved than proving the above result on degree directly. We will, therefore, use the classical definition of the degree of a map due to L.E.J. Brouwer, which is more intuitive than the "homology definition" given earlier, to prove the above result on degree. Indeed, Brouwer's definition of degree of a map $f\colon \mathbb{S}^n \to \mathbb{S}^n$ really means the number of times the domain sphere "wraps" around the range sphere. We have

Definition 4.8.3. *Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map and let $h\colon |K| \to \mathbb{S}^n$ be a triangulation of $\mathbb{S}^n$. Let $\phi\colon K^{(k)} \to K$ be a simplicial approximation of f, where $K^{(k)}$ is the kth barycentric subdivision of K. For any positively oriented n-simplex τ of K, let p be the number of positively oriented simplexes σ of $K^{(k)}$ such that $\phi(\sigma) = \tau$ and let q be the number of negatively oriented simplexes σ of $K^{(k)}$ such that $\phi(\sigma) = \tau$. Then the integer $p - q$ is independent of the choice of τ, K, $K^{(k)}$ and ϕ. This integer is called the* **degree** *of the map f.*

It is true that the two definitions of the degree of a map are equivalent, but we will not prove this fact here. The curious reader may like to see (Hocking and Young [10]) for a detailed discussion and proofs of all the "independent statements" made above.

It follows from Brouwer's definition that the map $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ defined by $f(z) = z^n$ has degree n; the degree of a constant map $f\colon \mathbb{S}^n \to \mathbb{S}^n, n \geq 1$, is zero; the degree of identity map $I_{\mathbb{S}^n}\colon \mathbb{S}^n \to \mathbb{S}^n$ is one, etc. Now, we can prove

Theorem 4.8.4. *If two continuous maps $f,\ g\colon \mathbb{S}^n \to \mathbb{S}^n$ are homotopic, then* $\deg f = \deg g$.

Proof. Let $H\colon \mathbb{S}^n \times I \to \mathbb{S}^n$ be a homotopy starting from f and terminating into g. If we put $H(x,t) = h_t(x)$, $x \in \mathbb{S}^n, t \in I$, then each $h_t\colon \mathbb{S}^n \to \mathbb{S}^n$ is a continuous map and $h_0 = f, h_1 = g$. Now, it suffices to show that the map $I \to \mathbb{Z}$ defined by $t \rightsquigarrow \deg h_t$ is constant. The last fact will follow if we can prove that the map from $I \to \mathbb{Z}$ is continuous because I is connected and $\mathbb{Z}$ has the discrete topology. This is what we now proceed to do: Let K be a triangulation of $\mathbb{S}^n$, and consider the open cover $\{\text{ost}(w_i) : w_i \text{ is a vertex of } K\}$. Suppose ϵ is a Lebesgue number for the open cover. Since H is uniformly continuous, we can find a positive real δ such that $A \subset \mathbb{S}^n, B \subset I$ with $\text{diam}(A) < \delta$, $\text{diam}(B) < \delta$ imply that $\text{diam}(H(A \times B)) < \epsilon$. Let $K^{(k)}$ be a barycentric subdivision of K with mesh less than $\delta/2$ so that for each vertex v of $K^{(k)}$, $\text{diam}(\text{ost}(v)) < \delta$. Select a partition

$$0 = t_0 < t_1 < \ldots < t_q = 1$$

of I for which $|t_j - t_{j-1}| < \delta, \forall j = 0, 1, \ldots, q$. Then for each vertex v_i of K, the set $h(\text{ost}(v_i) \times [t_{j-1}, t_j])$ has diameter less than ϵ, for each j. Therefore, there is a vertex w_{ij} of K such that $h(\text{ost}(v_i) \times [t_{j-1}, t_j]) \subseteq \text{ost}(w_{ij})$. Hence, for each $t \in [t_{j-1}, t_j]$, we can define a simplicial map ϕ_t by putting $\phi_t(v_i) = w_{ij}$, which is evidently a simplicial approximation to h_t, for all $t \in [t_{j-1}, t_j]$. Since ϕ_t is same for all $t \in [t_{j-1}, t_j]$, it follows from the Brouwer's definition of the degree of a map that h_t has the same degree for all $t \in [t_{j-1}, t_j]$, showing that $t \rightsquigarrow \deg h_t$ is a continuous map from I to $\mathbb{Z}$. ■

We must point out here that the converse of the above theorem was also studied by Brouwer. In fact, he had proved that if $f,\ g\colon \mathbb{S}^2 \to \mathbb{S}^2$ are two continuous maps such that $\deg f = \deg g$, then f and g are homotopic. It was H. Hopf (1894-1971) who proved the converse in full generality in the year 1927. We are not including the proof of the converse here, but give only the statement of this beautiful theorem:

Theorem 4.8.5. (Hopf's Classification Theorem). *Two continuous maps $f,\ g\colon \mathbb{S}^n \to \mathbb{S}^n$ are homotopic if and only if they have the same degree.*

We will remain contented only by stating the above theorem, though its several generalizations are also known. The most important aspect of the above theorem is to observe that the homotopy classes of maps from $\mathbb{S}^n$ to itself are completely classified by the set of integers! Since we are leaving the above theorem without giving its proof, we can as well state the following theorem without proof which is regarded as one of the most satisfying theorems of topology because it accomplishes the complete classification of compact 2-manifolds without boundary (called **closed 2-manifolds or closed surfaces**) up to homeomorphisms.

Theorem 4.8.6. *Two closed surfaces are homeomorphic if and only if they have the same Betti numbers in all dimensions.*

The above theorem says that any two closed 2-manifolds X and Y are homeomorphic if and only if $H_q(X) \cong H_q(Y), \forall q \geq 0$. This is one instance where the homology groups provide complete classification of closed 2-manifolds. Note that we have defined homology groups only for those compact topological spaces which are triangulable, i.e., are polyhedra. Hence it is natural to ask as to which manifolds are triangulable? Poincaré had asked this question for any n-manifold, $n \geq 2$, not necessarily compact – there is a suitable definition of simplicial complexes which are not necessarily finite. It is now well known through the works of several mathematicians that every 2-manifold, as well as every 3-manifold, is triangulable. Hence a surface is always triangulable whether it is compact or not. This is the reason that in the above theorem triangulability of the surfaces is not a part of the hypothesis. We must also mention here that the above theorem really shows the power of homology groups. In fact, this theorem was known as early as 1890 through the works of C. Jordan (1858–1922) and A.F. Möbius (1790–1860). As we know, C. Jordan is known for his "Jordan Curve" Theorem, and A.F. Möbius is known for his "Möbius band" through which he introduced the idea of "orientability". The modern definition of orientability in terms of homology groups was introduced by J.W. Alexander (1888–1971). A complete proof of the above theorem in a more precise form can be found in W.H. Massey [13].

Definition 4.8.7. *Let $\mathbb{S}^n$ be the unit n-sphere embedded in $\mathbb{R}^{n+1}$. Then the map $A\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $A(x) = -x$, $x \in \mathbb{S}^n$ is called the* **antipodal map**.

Note that the antipodal map is induced by a linear transformation $T\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ whose determinant is $(-1)^{n+1}$. The following result is, therefore, quite interesting.

Theorem 4.8.8. *The degree of the antipodal map $A\colon \mathbb{S}^n \to \mathbb{S}^n$ is $(-1)^{n+1}$, $n \geq 1$.*

Proof. We know that $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_1^{n+1} x_i^2 = 1\}$. A map $r_i\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $r_i(x_1, \ldots, x_i, \ldots, x_{n+1}) = (x_1, \ldots, -x_i, \ldots, x_{n+1})$ is called a **reflection map**. We will show that the map $r\colon \mathbb{S}^n \to \mathbb{S}^n$ defined by $r(x_1, \ldots, x_{n+1}) = (x_1, \ldots, -x_{n+1})$ has degree -1. This will mean that each reflection r_i has degree -1. To see this, note that if h is the homeomorphism of $\mathbb{S}^n$ which interchanges the coordinates x_i and x_{n+1} for a fixed i, then $h^{-1}rh = r_i$ and so $\deg r_i = \deg h^{-1} \cdot \deg r \cdot \deg h = \deg r$, proving that any reflection has degree (-1). Now, since the antipodal map $A = r_1 \circ r_2 \circ \ldots r_{n+1}$, it follows that $\deg A = (-1)^{n+1}$. Next, we are going to determine a triangulation $S(K)$ of $\mathbb{S}^n$ and a simplicial map $g\colon S(K) \to S(K)$ which will induce the map r. To do this, let K be a triangulation of $\mathbb{S}^{n-1}$ embedded in $\mathbb{S}^n$ in the standard way. Let w_0 and w_1 be two distinct vertices of the cones $w_0 * K$ and $w_1 * K$. Put $S(K) = (w_0 * K) \cup (w_1 * K)$. Let g be the simplicial map from $S(K)$ to itself which interchanges the vertices w_0 and w_1 and keeps all other vertices fixed. Let $h\colon |K| \to \mathbb{S}^{n-1}$ be a triangulating homeomorphism. We define $k\colon |S(K)| \to \mathbb{S}^n$ by the following formula: If $y = (1-t)x + tw_0$ for some $x \in |K|$, then define

$$k(y) = (\sqrt{1-t^2}\, h(x), t),$$

and if $y = (1-t)x + tw_1$, then put

$$k(y) = (\sqrt{1-t^2}\, h(x), -t).$$

Then k is a homeomorphism making the diagram

$$\begin{array}{ccc} |S(K)| & \xrightarrow{\quad k \quad} & \mathbb{S}^n \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle r} \\ |S(K)| & \xrightarrow[\quad k \quad]{} & \mathbb{S}^n \end{array}$$

commutative. Hence it suffices to show that $\deg g = -1$. Let z be an n-cycle of $S(K)$. Then z is a chain of the form

$$z = [w_0, c_m] + [w_1, d_m],$$

where c_m and d_m are chains in K and $m = n-1$. Assume $n > 1$. Since z is a cycle, we have

$$0 = \partial(z) = c_m - [w_0, \partial c_m] + d_m - [w_1, \partial d_m].$$

Restricting this chain to K, we get

$$0 = c_m + d_m.$$

Therefore,

$$z = [w_0, c_m] - [w_1, c_m].$$

Since g simply exchanges w_0, w_1, we find that

$$g_*(z) = [w_1, c_m] - [w_0, c_m] = -z.$$

Hence, $\deg g = -1$. The case $n = 1$ is proved similarly. ■

Tangent Vector Fields on Spheres

Now, we present an interesting application of the above theorem on the existence of tangent vector fields on Euclidean spheres. We have

Definition 4.8.9. *A continuous mapping $f\colon \mathbb{S}^n \to \mathbb{S}^n$ is said to be a* **tangent vector field** *on $\mathbb{S}^n$ if for each vector $x \in \mathbb{S}^n$, the two vectors x and $f(x)$ are perpendicular to each other.*